連立積分法について

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はじめに

積分の計算ほど技巧を要するものはない。ここでは，有理関数の積分のうち，とくに分数関数の積分について述べてみたいと思います。

§1．分数関数の積分（Ⅰ）

(イ)　分母が1次式のとき

$$\int\frac{dx}{ax+b}=\frac{1}{a}\log|ax+b|+C \quad (C：積分定数)$$

(ロ)　分母が2次式のとき

$$\int\frac{dx}{ax^2+bx+e}$$

$z=2ax+b$
$D=b^2-4ae$　とおくと

$$\begin{aligned}&4a(ax^2+bx+e)\\&=4a^2x^2+4abx+4ae\\&=(2ax+b)^2-(b^2-4ae)\\&=z^2-D\end{aligned}$$

(1)　$D>0$ のとき

$$\int\frac{dx}{ax^2+bx+e}=2\int\frac{dz}{z^2-D}=\frac{1}{\sqrt{D}}\log\left|\frac{z-\sqrt{D}}{z+\sqrt{D}}\right|+C$$

(2)　$D=0$ のとき

$$\int\frac{dx}{ax^2+bx+e}=2\int\frac{dz}{z^2}=-\frac{2}{z}+C$$

(3)　$D<0$ のとき

$$\int\frac{dx}{ax^2+bx+e}=2\int\frac{dz}{z^2-D}=\frac{2}{\sqrt{-D}}\tan^{-1}\frac{z}{\sqrt{-D}}+C$$

§2．分数関数の積分（Ⅱ）

分母が2次式で，かんたんに因数分解ができるもの，また，無理式（1次）を含めて，従来の解法と[別解]として，連立積分法で解きます。

(1)　$\int\frac{x}{(x+2)^2}dx$（置き換える方法）

$x+2=t$ とおくと，$x=t-2$，$dx=dt$ より

$$\begin{aligned}\int\frac{x}{(x+2)^2}dx&=\int\frac{t-2}{t^2}dt=\int\left(\frac{1}{t}-\frac{2}{t^2}\right)dt\\&=\log|t|+\frac{2}{t}+C\\&=\log|x+2|+\frac{2}{x+2}+C\end{aligned}$$

[別解]

$\int\frac{dx}{(x+2)^2}=u$，$\int\frac{x}{(x+2)^2}dx=v$ とおく。

$$u=\int\frac{dx}{(x+2)^2}=-\frac{1}{x+2}\quad\cdots\cdots①$$

$$\begin{aligned}2u+v&=\int\frac{2}{(x+2)^2}dx+\int\frac{x}{(x+2)^2}dx=\int\frac{dx}{(x+2)}\\&=\log|x+2|\quad\cdots\cdots②\end{aligned}$$

②−①×2　$v=\log|x+2|+\frac{2}{x+2}$

$$\therefore\quad\int\frac{x}{(x+2)^2}dx=\log|x+2|+\frac{2}{x+2}+C$$

(2)　$\int\frac{x}{x^2-6x+8}dx$（部分分数に分解する方法）

$\frac{x}{x^2-6x+8}=\frac{A}{x-2}+\frac{B}{x-4}$ より $A=-1$，$B=2$

$$\begin{aligned}\int\frac{x}{x^2-6x+8}dx&=\int\left(\frac{2}{x-4}-\frac{1}{x-2}\right)dx\\&=2\log|x-4|-\log|x-2|+C\\&=\log\frac{(x-4)^2}{|x-2|}+C\end{aligned}$$

[別解]

$\int\frac{dx}{(x-2)(x-4)}=u$，$\int\frac{x}{(x-2)(x-4)}dx=v$

とおく。

$$\begin{aligned}v-4u&=\int\frac{x-4}{(x-2)(x-4)}dx\\&=\int\frac{dx}{x-2}=\log|x-2|\quad\cdots\cdots①\end{aligned}$$

$$\begin{aligned}v-2u&=\int\frac{x-2}{(x-2)(x-4)}dx\\&=\int\frac{dx}{x-4}=\log|x-4|\quad\cdots\cdots②\end{aligned}$$

②×2−①　$v=2\log|x-4|-\log|x-2|$

$$\therefore\ \int\frac{x}{x^2-6x+8}dx=\log\frac{(x-4)^2}{|x-2|}+C$$

(3) $\displaystyle\int\frac{2x-3}{x^2-3x+2}dx$ (微分する方法)

$(x^2-3x+2)'=2x-3$ より

$$\int\frac{2x-3}{x^2-3x+2}dx=\log|x^2-3x+2|+C$$

[別解]

$\displaystyle\int\frac{dx}{x^2-3x+2}=u,\ \int\frac{x}{x^2-3x+2}dx=v$ とおく。

$x^2-3x+2=(x-1)(x-2)$ より

$$-u+v=\int\frac{x-1}{(x-1)(x-2)}dx$$

$$=\int\frac{dx}{x-2}=\log|x-2|\quad\cdots\cdots①$$

$$-2u+v=\int\frac{x-2}{(x-1)(x-2)}dx$$

$$=\int\frac{dx}{x-1}=\log|x-1|\quad\cdots\cdots②$$

①+②　$-3u+2v=\log|x-2|+\log|x-1|$

$$\therefore\ \int\frac{2x-3}{x^2-3x+2}dx=\log|x^2-3x+2|+C$$

(4) $\displaystyle\int\frac{2x}{\sqrt{4x+1}}dx$ (置き換える方法)

$4x+1=t$ とおくと，$x=\frac{1}{4}(t-1)$，$dx=\frac{1}{4}dt$

より

$$\int\frac{2x}{\sqrt{4x+1}}dx=\int\frac{1}{\sqrt{t}}\cdot\frac{t-1}{2}\cdot\frac{1}{4}dt$$

$$=\int\frac{1}{8}\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)dt$$

$$=\frac{1}{8}\left(\frac{2}{3}t\sqrt{t}-2\sqrt{t}\right)+C$$

$$=\frac{1}{6}(2x-1)\sqrt{4x+1}+C$$

[別解]

$\displaystyle\int\frac{dx}{\sqrt{4x+1}}=u,\ \int\frac{x}{\sqrt{4x+1}}dx=v$ とおくと

$$u+4v=\int\frac{dx}{\sqrt{4x+1}}+\int\frac{4x}{\sqrt{4x+1}}dx=\int\sqrt{4x+1}\,dx$$

$$=\frac{1}{6}(4x+1)^{\frac{3}{2}}\quad\cdots\cdots①$$

$$u=\int\frac{dx}{\sqrt{4x+1}}=\frac{1}{2}(4x+1)^{\frac{1}{2}}\quad\cdots\cdots②$$

①−②　$4v=\frac{1}{6}(4x+1)^{\frac{3}{2}}-\frac{1}{2}(4x+1)^{\frac{1}{2}}$

$$=\frac{1}{6}\{(4x+1)\sqrt{4x+1}-3\sqrt{4x+1}\}$$

$$=\frac{2}{6}(2x-1)\sqrt{4x+1}$$

$$\therefore\ \int\frac{2x}{\sqrt{4x+1}}dx=\frac{1}{6}(2x-1)\sqrt{4x+1}+C$$

§3. 連立積分法(Ⅰ)(分母が3次式)

(1) $\displaystyle\int\frac{dx}{x^3-1},\ \int\frac{x}{x^3-1}dx,\ \int\frac{x^2}{x^3-1}dx$ を求めよ。

$\displaystyle\int\frac{dx}{x^3-1}=u,\ \int\frac{x}{x^3-1}dx=v,\ \int\frac{x^2}{x^3-1}=w$

とおくと

$$u+v+w=\int\frac{x^2+x+1}{x^3-1}dx=\int\frac{dx}{x-1}=\log|x-1|$$

また

$$w=\int\frac{x^2}{x^3-1}dx=\frac{1}{3}\int\frac{3x^2}{x^3-1}dx=\log|x^3-1|$$

より

$$u+v=\log|x-1|-\frac{1}{3}\log|x^3-1|\quad\cdots\cdots①$$

$$-u+v=\int\frac{x-1}{x^3-1}dx=\int\frac{dx}{x^2+x+1}$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}\quad\cdots\cdots②$$

①+②

$$2v=\log|x-1|-\frac{1}{3}\log|x^3-1|+\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$$

①−②

$$2u=\log|x-1|-\frac{1}{3}\log|x^3-1|-\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$$

$$\therefore\ \int\frac{dx}{x^3-1}$$

$$=\frac{1}{6}\log\frac{(x-1)^2}{x^2+x+1}-\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+C$$

$$\int\frac{x}{x^3-1}dx$$

$$=\frac{1}{6}\log\frac{(x-1)^2}{x^2+x+1}+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+C$$

$$\int\frac{x^2}{x^3-1}dx=\frac{1}{3}\log|x^3-1|+C$$

(2) $\displaystyle\int\frac{dx}{x^3+1},\ \int\frac{x}{x^3+1}dx,\ \int\frac{x^2}{x^3+1}dx$ を求めよ。

$\int\frac{dx}{x^3+1}=u$, $\int\frac{x}{x^3+1}dx=v$, $\int\frac{x^2}{x^3+1}dx=w$

とおくと

$$u-v+w=\int\frac{x^2-x+1}{x^3+1}dx=\int\frac{dx}{x+1}=\log|x+1|$$

また

$$w=\int\frac{x^2}{x^3+1}dx=\frac{1}{3}\int\frac{3x^2}{x^3+1}dx=\frac{1}{3}\log|x^3+1|$$

より

$$u-v=\log|x+1|-\frac{1}{3}\log|x^3+1| \quad \cdots\cdots①$$

$$u+v=\int\frac{x+1}{x^3+1}dx=\int\frac{dx}{x^2-x+1}$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} \quad \cdots\cdots②$$

①+②

$$2u=\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}+\log|x+1|-\frac{1}{3}\log|x^3+1|$$

②−①

$$2v=\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}-\log|x+1|+\frac{1}{3}\log|x^3+1|$$

$\therefore$ $\int\frac{dx}{x^3+1}$

$$=\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}+\frac{1}{6}\log\frac{(x+1)^2}{x^2-x+1}+C$$

$\int\frac{x}{x^3+1}dx$

$$=\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}-\frac{1}{6}\log\frac{(x+1)^2}{x^2-x+1}+C$$

$$\int\frac{x^2}{x^3+1}dx=\frac{1}{3}\log|x^3+1|+C$$

(3) $\int\frac{x^2-3}{x(x^2-1)}dx$ を求めよ。

$\int\frac{dx}{x(x^2-1)}=u$, $\int\frac{x}{x(x^2-1)}dx=v$,

$\int\frac{x^2}{x(x^2-1)}dx=w$ とおくと

$$-u+v=\int\frac{x-1}{x(x^2-1)}dx$$

$$=\int\frac{dx}{x(x+1)}=\int\left(\frac{1}{x}-\frac{1}{x+1}\right)dx$$

$$=\log|x|-\log|x+1| \quad \cdots\cdots①$$

$$u+v=\int\frac{x+1}{x(x^2-1)}dx$$

$$=\int\frac{dx}{x(x-1)}=\int\left(\frac{1}{x-1}-\frac{1}{x}\right)dx$$

$$=\log|x-1|-\log|x| \quad \cdots\cdots②$$

$$-u+w=\int\frac{x^2-1}{x(x^2-1)}dx$$

$$=\int\frac{dx}{x}=\log|x| \quad \cdots\cdots③$$

①−②+③

$$-3u+w=3\log|x|-\log|x+1|-\log|x-1|$$

$$\therefore \int\frac{x^2-3}{x(x^2-1)}dx=\log\left|\frac{x^3}{x^2-1}\right|+C$$

(4) $\int\frac{2x}{(x+1)(x^2+1)}dx$ を求めよ。

$\int\frac{dx}{(x+1)(x^2+1)}=u$, $\int\frac{x}{(x+1)(x^2+1)}dx=v$,

$\int\frac{x^2}{(x+1)(x^2+1)}dx=w$ とおくと

$$u+v=\int\frac{x+1}{(x+1)(x^2+1)}dx$$

$$=\int\frac{dx}{x^2+1}=\tan^{-1}x \quad \cdots\cdots①$$

$$v+w=\int\frac{x^2+x}{(x+1)(x^2+1)}dx$$

$$=\int\frac{x}{x^2+1}dx=\frac{1}{2}\log(x^2+1) \quad \cdots\cdots②$$

$$w+u=\int\frac{x^2+1}{(x+1)(x^2+1)}dx$$

$$=\int\frac{dx}{x+1}=\log|x+1| \quad \cdots\cdots③$$

①+②−③

$$2v=\tan^{-1}x+\frac{1}{2}\log(x^2+1)-\log|x+1|$$

$\therefore$ $\int\frac{2x}{(x+1)(x^2+1)}dx$

$$=\tan^{-1}x+\log\frac{\sqrt{x^2+1}}{|x+1|}+C$$

(5) $\int\frac{2}{x(x^2+1)}dx$ を求めよ。

$\int\frac{dx}{x(x^2+1)}=u$, $\int\frac{x}{x(x^2+1)}dx=v$,

$\int\frac{x^2}{x(x^2+1)}dx=w$ とおくと

$$u+w=\int\frac{dx}{x(x^2+1)}+\int\frac{x^2}{x(x^2+1)}dx=\int\frac{dx}{x}$$

$$=\log|x| \quad \cdots\cdots①$$

$$v+3w=\int\frac{dx}{x(x^2+1)}+\int\frac{3x^2}{x(x^2+1)}dx$$

$$=\int\frac{3x^2+1}{x(x^2+1)}dx=\int\frac{(x^3+x)'}{x^3+x}dx$$

$$=\log|x^3+x| \quad \cdots\cdots②$$

①×3−②　$2u=3\log|x|-\log|x(x^2+1)|$

$$\therefore \int\frac{2}{x(x^2+1)}dx=\log\frac{x^2}{x^2+1}+C$$

(6) $\displaystyle\int\frac{2x+1}{(x-1)(x+2)^2}dx$ を求めよ。

$\displaystyle\int\frac{dx}{(x-1)(x+2)^2}=u$, $\displaystyle\int\frac{x}{(x-1)(x+2)^2}dx=v$,

$\displaystyle\int\frac{x^2}{(x-1)(x+2)^2}dx=v$ とおくと

$$-u+v=\int\frac{x-1}{(x-1)(x+2)^2}dx$$

$$=\int\frac{dx}{(x+2)^2}=\frac{-1}{x+2} \quad \cdots\cdots①$$

$$4u+4v+w=\int\frac{x^2+4x+4}{(x-1)(x+2)^2}dx$$

$$=\int\frac{dx}{x-1}=\log|x-1| \quad \cdots\cdots②$$

$$-2u+v+w=\int\frac{x^2+x-2}{(x-1)(x+2)^2}dx$$

$$=\int\frac{dx}{x+2}=\log|x+2| \quad \cdots\cdots③$$

②−③　$6u+3v=\log|x-1|-\log|x+2|$

$$2u+v=\frac{1}{3}(\log|x-1|-\log|x+2|) \quad \cdots\cdots④$$

①+④

$$u+2v=-\frac{1}{x+2}+\frac{1}{3}(\log|x-1|-\log|x+2|)$$

$$\therefore \int\frac{2x+1}{(x-1)(x+2)^2}dx$$

$$=\frac{1}{3}\log\left|\frac{x-1}{x+2}\right|-\frac{1}{x+2}+C$$

§4. 連立積分法（Ⅱ）（分母が４次式）

(1) $\displaystyle\int\frac{dx}{x^4-1}$, $\displaystyle\int\frac{x}{x^4-1}dx$, $\displaystyle\int\frac{x^2}{x^4-1}dx$, $\displaystyle\int\frac{x^3}{x^4-1}dx$ を求めよ。

$\displaystyle\int\frac{dx}{x^4-1}=u$, $\displaystyle\int\frac{x}{x^4-1}dx=v$, $\displaystyle\int\frac{x^2}{x^4-1}dx=w$,

$\displaystyle\int\frac{x^3}{x^4-1}dx=s$ とおくと

$$u+w=\int\frac{x^2+1}{x^4-1}dx=\int\frac{dx}{x^2-1}$$

$$=\frac{1}{2}(\log|x-1|-\log|x+1|) \quad \cdots\cdots①$$

$$-u+w=\int\frac{x^2-1}{x^4-1}dx=\int\frac{dx}{x^2+1}$$

$$=\tan^{-1}x \quad \cdots\cdots②$$

$$v+s=\int\frac{x^3+x}{x^4-1}dx=\int\frac{x}{x^2-1}dx$$

$$=\frac{1}{2}\log|x^2-1| \quad \cdots\cdots③$$

$$-v+s=\int\frac{x^3-x}{x^4-1}dx=\int\frac{x}{x^2+1}dx$$

$$=\frac{1}{2}\log|x^2+1| \quad \cdots\cdots④$$

①−②　$2u=\frac{1}{2}\log|x-1|-\frac{1}{2}\log|x+1|-\tan^{-1}x$

①+②　$2w=\frac{1}{2}\log|x-1|-\frac{1}{2}\log|x+1|+\tan^{-1}x$

③−④　$2v=\frac{1}{2}\log|x^2-1|-\frac{1}{2}\log|x^2+1|$

③+④　$2s=\frac{1}{2}\log|x^2-1|+\frac{1}{2}\log|x^2+1|$

$$\therefore \int\frac{dx}{x^4-1}=\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x+C$$

$$\int\frac{x}{x^4-1}dx=\frac{1}{4}\log\left|\frac{x^2-1}{x^2+1}\right|+C$$

$$\int\frac{x^2}{x^4-1}dx=\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|+\frac{1}{2}\tan^{-1}x+C$$

$$\int\frac{x^3}{x^4-1}dx=\frac{1}{4}\log|x^4-1|+C$$

(2) $\displaystyle\int\frac{dx}{x^4+1}$, $\displaystyle\int\frac{x}{x^4+1}dx$, $\displaystyle\int\frac{x^2}{x^4+1}dx$, $\displaystyle\int\frac{x^3}{x^4+1}dx$ を求めよ。

$\displaystyle\int\frac{dx}{x^4+1}=u$, $\displaystyle\int\frac{x}{x^4+1}dx=v$, $\displaystyle\int\frac{x^2}{x^4+1}dx=w$,

$\displaystyle\int\frac{x^3}{x^4+1}dx=s$ とおくと

$$u+w=\int\frac{x^2+1}{x^4+1}dx$$

$$=\frac{1}{2}\int\left(\frac{1}{x^2-\sqrt{2}x+1}+\frac{1}{x^2+\sqrt{2}x+1}\right)dx$$

$$=\frac{1}{2}\left(\frac{2}{\sqrt{2}}\tan^{-1}\frac{2x-\sqrt{2}}{\sqrt{2}}+\frac{2}{\sqrt{2}}\tan^{-1}\frac{2x+\sqrt{2}}{\sqrt{2}}\right)$$

$$=\frac{1}{\sqrt{2}}\{\tan^{-1}(\sqrt{2}x-1)+\tan^{-1}(\sqrt{2}x+1)\}$$

ここで，$\tan^{-1}(\sqrt{2}x-1)=\alpha$，$\tan^{-1}(\sqrt{2}x+1)=\beta$ とおくと $\tan\alpha=\sqrt{2}x-1$，$\tan\beta=\sqrt{2}x+1$ より

$\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\dfrac{\sqrt{2}x}{1-x^2}$ となる。

$\alpha+\beta=\tan^{-1}\dfrac{\sqrt{2}x}{1-x^2}$ より

$$u+w=\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{2}x}{1-x^2} \quad \cdots\cdots①$$

$$-u+w=\int\frac{x^2-1}{x^4+1}dx=\int\frac{\left(x-\frac{1}{x}\right)x}{\left(x^2+\frac{1}{x^2}\right)x^2}dx$$

$$=\int\frac{x-\frac{1}{x}}{x^2+\frac{1}{x^2}}\cdot\frac{dx}{x}$$

$x+\dfrac{1}{x}=t$ とおくと

$\left(x-\dfrac{1}{x}\right)\dfrac{dx}{x}=dt$, $x^2+\dfrac{1}{x^2}=t^2-2$ より

$$=\int\frac{dt}{t^2-2}=\frac{1}{2\sqrt{2}}\log\frac{t-\sqrt{2}}{t+\sqrt{2}} \quad \cdots\cdots②$$

①+② $2w=\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{\sqrt{2}x}{1-x^2}+\dfrac{1}{2\sqrt{2}}\log\dfrac{t-\sqrt{2}}{t+\sqrt{2}}$

①−② $2u=\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{\sqrt{2}x}{1-x^2}-\dfrac{1}{2\sqrt{2}}\log\dfrac{t-\sqrt{2}}{t+\sqrt{2}}$

また $x^2=y$ とおくと $dy=2x\,dx$ より

$$\int\frac{x}{x^4+1}dx=\frac{1}{2}\int\frac{2x}{(x^2)^2+1}dx$$

$$=\frac{1}{2}\int\frac{dy}{y^2+1}=\frac{1}{2}\tan^{-1}y$$

$x^4=y$ とおくと $dy=4x^3\,dx$ より

$$\int\frac{x^3}{x^4+1}dx=\frac{1}{4}\int\frac{4x^3}{x^4+1}dx=\frac{1}{4}\int\frac{dy}{y+1}$$

$$=\frac{1}{4}\log|y+1|$$

$\therefore\ \displaystyle\int\frac{dx}{x^4+1}$

$$=\frac{1}{2\sqrt{2}}\tan^{-1}\frac{\sqrt{2}x}{1-x^2}-\frac{1}{4\sqrt{2}}\log\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}+C$$

$$\int\frac{x}{x^4+1}dx=\frac{1}{2}\tan^{-1}x^2+C$$

$\displaystyle\int\frac{x^2}{x^4+1}dx$

$$=\frac{1}{2\sqrt{2}}\tan^{-1}\frac{\sqrt{2}x}{1-x^2}+\frac{1}{4\sqrt{2}}\log\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}+C$$

$$\int\frac{x^3}{x^4+1}dx=\frac{1}{4}\log(x^4+1)+C$$

(3) $\displaystyle\int\frac{3x}{(x-1)(x^3-1)}dx$ を求めよ。

$\displaystyle\int\frac{dx}{(x-1)(x^3-1)}=u$, $\displaystyle\int\frac{x}{(x-1)(x^3-1)}dx=v$,

$\displaystyle\int\frac{x^2}{(x-1)(x^3-1)}dx=w$ とおくと

$$u+v+w=\int\frac{x^2+x+1}{(x-1)(x^3-1)}dx=\int\frac{dx}{(x-1)^2}$$

$$=-\frac{1}{x-1} \quad \cdots\cdots①$$

$$u-2v+w=\int\frac{x^2-2x+1}{(x-1)(x^3-1)}dx=\int\frac{dx}{x^2+x+1}$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} \quad \cdots\cdots②$$

①−② $3v=-\dfrac{1}{x-1}-\dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{2x+1}{\sqrt{3}}$

$\therefore\ \displaystyle\int\frac{3x}{(x-1)(x^3-1)}dx$

$$=-\frac{1}{x-1}-\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+C$$

(4) $\displaystyle\int\frac{dx}{(x^2+1)(x^2+4)}$ を求めよ。

$\displaystyle\int\frac{dx}{(x^2+1)(x^2+4)}=u$, $\displaystyle\int\frac{x}{(x^2+1)(x^2+4)}dx=v$,

$\displaystyle\int\frac{x^2}{(x^2+1)(x^2+4)}dx=w$ とおくと

$$u+w=\int\frac{x^2+1}{(x^2+1)(x^2+4)}dx=\int\frac{dx}{x^2+4}$$

$$=\frac{1}{2}\tan^{-1}\frac{x}{2} \quad \cdots\cdots①$$

$$4u+w=\int\frac{x^2+4}{(x^2+1)(x^2+4)}dx=\int\frac{dx}{x^2+1}$$

$$=\tan^{-1}x \quad \cdots\cdots②$$

②−① $3u=\tan^{-1}x-\dfrac{1}{2}\tan^{-1}\dfrac{x}{2}$

$$\therefore\ \int\frac{dx}{(x^2+1)(x^2+4)}=\frac{1}{3}\tan^{-1}x-\frac{1}{6}\tan^{-1}\frac{x}{2}+C$$

(5) $\displaystyle\int\frac{dx}{(x^2-1)^2}$ を求めよ。

$\displaystyle\int\frac{dx}{(x^2-1)^2}=u$, $\displaystyle\int\frac{x}{(x^2-1)^2}dx=v$,

$\displaystyle\int\frac{x^2}{(x^2-1)^2}dx=w$ とおくと

$$-u+w=\int\frac{x^2-1}{(x^2-1)^2}dx=\int\frac{dx}{x^2-1}$$

$$=\frac{1}{2}\log\left|\frac{x-1}{x+1}\right| \quad \cdots\cdots①$$

$$u-2v+w=\int\frac{x^2-2x+1}{(x^2-1)^2}dx$$

$$=\int\frac{dx}{(x+1)^2}=-\frac{1}{x+1} \quad \cdots\cdots ②$$

$$u+2v+w=\int\frac{x^2+2x+1}{(x^2-1)^2}dx$$

$$=\int\frac{dx}{(x-1)^2}=-\frac{1}{x-1} \quad \cdots\cdots ③$$

②+③ $2u+2w=-\dfrac{1}{x+1}-\dfrac{1}{x-1}$

$$u+w=-\frac{1}{2}\left(\frac{1}{x+1}+\frac{1}{x-1}\right) \quad \cdots\cdots ④$$

④−① $2u=-\dfrac{1}{2}\left(\dfrac{1}{x+1}+\dfrac{1}{x-1}\right)-\dfrac{1}{2}\log\left|\dfrac{x-1}{x+1}\right|$

$$\therefore \int\frac{dx}{(x^2-1)^2}=-\frac{1}{4}\left(\frac{2x}{x^2-1}+\log\left|\frac{x-1}{x+1}\right|\right)+C$$

(6) $\displaystyle\int\frac{6x}{(x^2-1)(x^2-4)}dx$ を求めよ。

$\displaystyle\int\frac{dx}{(x^2-1)(x^2-4)}=u$, $\displaystyle\int\frac{x}{(x^2-1)(x^2-4)}dx=v$,

$\displaystyle\int\frac{x^2}{(x^2-1)(x^2-4)}dx=w$ とおくと

$$2u+3v+w=\int\frac{x^2+3x+2}{(x^2-1)(x^2-4)}dx$$

$$=\int\frac{dx}{(x-1)(x-2)}=\log\left|\frac{x-2}{x-1}\right| \quad \cdots\cdots ①$$

$$2u-3v+w=\int\frac{x^2-3x+2}{(x^2-1)(x^2-4)}dx$$

$$=\int\frac{dx}{(x+1)(x+2)}=\log\left|\frac{x+1}{x+2}\right| \quad \cdots\cdots ②$$

①−② $6v=\log\left|\dfrac{x-2}{x-1}\right|-\log\left|\dfrac{x+1}{x+2}\right|$

$$\therefore \int\frac{6x}{(x^2-1)(x^2-4)}dx=\log\left|\frac{x^2-4}{x^2-1}\right|+C$$

(7) $\displaystyle\int\frac{x^3+x-1}{x^2(x^2+1)}dx$ を求めよ。

$\displaystyle\int\frac{dx}{x^2(x^2+1)}=u$, $\displaystyle\int\frac{x}{x^2(x^2+1)}dx=v$,

$\displaystyle\int\frac{x^2}{x^2(x^2+1)}dx=w$, $\displaystyle\int\frac{x^3}{x^2(x^2+1)}dx=s$ とおくと

$$w=\int\frac{x^2}{x^2(x^2+1)}dx=\int\frac{dx}{x^2+1}=\tan^{-1}x \quad \cdots\cdots ①$$

$$u+w=\int\frac{x^2+1}{x^2(x^2+1)}dx=\int\frac{dx}{x^2}=-\frac{1}{x} \quad \cdots\cdots ②$$

$$v+s=\int\frac{x^3+x}{x^2(x^2+1)}dx=\int\frac{dx}{x}=\log|x| \quad \cdots\cdots ③$$

①−②+③ $-u+v+s=\tan^{-1}x+\dfrac{1}{x}+\log|x|$

$$\therefore \int\frac{x^3+x-1}{x^2(x^2+1)}dx=\log|x|+\frac{1}{x}+\tan^{-1}x+C$$

§5. 連立積分法(Ⅲ)(分母が5次式以上)

(1) $\displaystyle\int\frac{dx}{x(x^3+1)^2}$ (置き換える方法)

$x^3=t$ とおくと，$3x^2\,dx=dt$ となり，

$\displaystyle\int\frac{3x^2}{3x^3(x^3+1)^2}dx=\frac{1}{3}\int\frac{dt}{t(t+1)^2}$ となる。

$\displaystyle\int\frac{dt}{t(t+1)^2}=u$, $\displaystyle\int\frac{t}{t(t+1)^2}dt=v$, $\displaystyle\int\frac{t^2}{t(t+1)^2}dt=w$

とおき $\displaystyle u+v=\int\frac{t+1}{t(t+1)^2}dt=\int\frac{dt}{t(t+1)}$

$$=\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt=\log|t|-\log|t+1| \quad \cdots\cdots ①$$

$$v=\int\frac{t}{t(t+1)^2}dt=\int\frac{dt}{(t+1)^2}=-\frac{1}{t+1} \quad \cdots\cdots ②$$

①−② $u=\log\left|\dfrac{t}{t+1}\right|+\dfrac{1}{t+1}$

$$\therefore \int\frac{dx}{x(x^3+1)^2}=\frac{1}{3}\left(\log\left|\frac{x^3}{x^3+1}\right|+\frac{1}{x^3+1}\right)+C$$

おわりに

連立積分法の解法(部分分数に分解する解法を含む)で，有理関数の分母が3次式，4次式の例をあげたが，次数が高くなると，連立方程式の文字数が多くなり，計算が複雑になるので，置換法等を併用した方が簡単になる。

また，分母が無理関数や超越関数の場合，可能な式もあるが，従来の解法がよいと思うから，この連立積分法による解法は，有理関数(分母が2・3・4次式で因数分解が可能)および置換法により，前記の式の形になるものに限られると思う。

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